Complex Numbers for JEE Main Preparation

Advanced Guide for Classes XI-XII

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In-Depth Theory, Hard-Level Examples, and Exercises
Prepared for JEE Main Aspirants

Chapter Synopsis

This booklet provides a rigorous study of "Complex Numbers" for JEE Main, with a focus on advanced concepts and challenging problems. It includes:

- **Comprehensive Theory**: Detailed explanations with geometric insights and proofs.
- **Hard-Level Examples**: JEE Main/Advanced-style problems with step-by-step solutions.
- **Pattern-Based Exercises**: High-difficulty problems, including MCQs, integer-type, and assertion-reason questions.
- **Applications**: Real-world and geometric applications.

1 Introduction to Complex Numbers

A complex number $z = a + bi$, where $a, b \in \mathbb{R}, i = \sqrt{-1}$, represents a point (a, b) in the complex plane. The real part is $\text{Re}(z) = a$, and the imaginary part is $\text{Im}(z) = b$.

1.1 Imaginary Unit

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^n = i^{n \bmod 4}$$

Equality: $a + bi = c + di$ if $a = c, b = d$.

Example 1 (Hard)

Solve for $x, y \in \mathbb{R}$: $(x + yi)^2 + 2(x - yi) = 5 + i$.
Solution: Let $z = x + yi$, so $\bar{z} = x - yi$. The equation becomes:

$$z^2 + 2\bar{z} = 5 + i$$

Compute $z^2 = (x + yi)^2 = x^2 - y^2 + 2xyi$, and $2\bar{z} = 2x - 2yi$. Equate real and imaginary parts:

$$x^2 - y^2 + 2x = 5, \quad 2xy - 2y = 1$$

Solving yields a quartic equation: $4t^4 + 16t^3 + 12t^2 - 1 = 0$.
Answer: Requires numerical solver; approximate solutions exist.

Example 2 (Hard)

Solve $(2x + yi)^2 = 9 + 40i$ for $x, y \in \mathbb{R}$.
Solution: Let $z = 2x + yi$. Then:

$$z^2 = (2x + yi)^2 = 4x^2 - y^2 + 4xyi = 9 + 40i$$

Equate parts: $4x^2 - y^2 = 9$, $4xy = 40$. Thus, $y = \frac{10}{x}$. Substitute:

$$4x^2 - \frac{100}{x^2} = 9 \implies 4x^4 - 100 = 9x^2 \implies 4x^4 - 9x^2 - 100 = 0$$

Solve: $x^2 = \frac{9 \pm \sqrt{81+1600}}{8} = 25, -\frac{16}{4}$. Thus, $x = \pm 5$, $y = \pm 2$.
Answer: $(x, y) = (5, 2), (-5, -2)$.

Example 3 (Hard)

Find $x, y \in \mathbb{R}$ such that $(x + yi) + (x - yi)^2 = 3 + 4i$.
Solution: Let $z = x + yi$, $\bar{z} = x - yi$. The equation is:

$$z + \bar{z}^2 = 3 + 4i$$

Since $\bar{z} = x - yi$, $\bar{z}^2 = (x - yi)^2 = x^2 - y^2 - 2xyi$. Then:

$$(x + yi) + (x^2 - y^2 - 2xyi) = (x + x^2 - y^2) + (y - 2xy)i = 3 + 4i$$

Equate: $x + x^2 - y^2 = 3$, $y - 2xy = 4$. From the second, $y(1 - 2x) = 4$. Solve numerically or test: $x = -1$, $y = -2$ satisfies.
Answer: $(x, y) = (-1, -2)$.

Example 4 (Hard)

Solve $(x + yi)^3 = 8i$ for $x, y \in \mathbb{R}$.
Solution: Let $z = x + yi$. Then $z^3 = 8i = 8e^{i\pi/2}$. The cube roots are:

$$z = 2e^{i(\pi/6 + 2k\pi/3)}, \quad k = 0, 1, 2$$

For $k = 1$, $\theta = \pi/6 + 2\pi/3 = 5\pi/6$, $z = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -\sqrt{3} + i$.
Answer: $(x, y) = (-\sqrt{3}, 1)$.

Example 5 (Hard)

Solve $(x + yi)^2 = x - yi$ for $x, y \in \mathbb{R}$.
Solution: Let $z = x + yi$. Then:

$$z^2 = \bar{z} \implies (x + yi)^2 = x - yi \implies x^2 - y^2 + 2xyi = x - yi$$

Equate: $x^2 - y^2 = x$, $2xy = -y$. From the second, $y(2x + 1) = 0$. If $y = 0$, then $x^2 = x \implies x = 0, 1$. If $2x + 1 = 0$, $x = -\frac{1}{2}$, then $y^2 = -\frac{1}{2}$, impossible. Solutions: $(0, 0)$, $(1, 0)$.
Answer: $(x, y) = (0, 0), (1, 0)$.

2 Arithmetic Operations on Complex Numbers

For $z_1 = a + bi$, $z_2 = c + di$:

- **Addition**: $z_1 + z_2 = (a + c) + (b + d)i$
- **Subtraction**: $z_1 - z_2 = (a - c) + (b - d)i$
- **Multiplication**: $z_1 z_2 = (ac - bd) + (ad + bc)i$
- **Division**: $\frac{z_1}{z_2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

Example 1 (Hard)

If $z = \frac{(1+i)^3}{(2-i)^2}$, find z.
Solution:

$$(1 + i)^3 = -2 + 2i, \quad (2 - i)^2 = 3 - 4i, \quad z = \frac{-2 + 2i}{3 - 4i} = -\frac{14}{25} - \frac{2}{25}i$$

Answer: $-\frac{14}{25} - \frac{2}{25}i$.

Example 2 (Hard)

Compute $\frac{(2+3i)^2}{(1+i)}$.
Solution:

$$(2+3i)^2 = 4+12i-9 = -5+12i, \quad \frac{-5 + 12i}{1 + i} \cdot \frac{1 - i}{1 - i} = \frac{(-5 + 12i)(1 - i)}{2} = \frac{-5 + 5i + 12i + 12}{2} = \frac{7 + 17i}{2}$$

Answer: $\frac{7}{2} + \frac{17}{2}i$.

Example 3 (Hard)

Find $\frac{(3-i)^3}{(2+i)^2}$.
Solution:

$$(3 - i)^3 = 27 - 27i + 9i^2 - i^3 = 18 - 26i, \quad (2 + i)^2 = 4 + 4i - 1 = 3 + 4i$$

$$\frac{18 - 26i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{(18 - 26i)(3 - 4i)}{25} = \frac{54 - 72i - 78i - 104}{25} = \frac{-50 - 150i}{25} = -2 - 6i$$

Answer: $-2 - 6i$.

Example 4 (Hard)

Compute $(1 + 2i)^2(3 - i)$.
Solution:

$$(1 + 2i)^2 = 1 + 4i - 4 = -3 + 4i, \quad (-3 + 4i)(3 - i) = -9 + 3i + 12i + 4 = -5 + 15i$$

Answer: $-5 + 15i$.

Example 5 (Hard)

Find $\frac{1+i}{2-3i} + \frac{2-i}{1+i}$.
Solution:

$$\frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{(1+i)(2+3i)}{13} = \frac{-1+5i}{13}, \quad \frac{2-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{(2-i)(1-i)}{2} = \frac{1-3i}{2}$$

$$\frac{-1+5i}{13} + \frac{1-3i}{2} = \frac{-2+10i+13-39i}{26} = \frac{11-29i}{26}$$

Answer: $\frac{11}{26} - \frac{29}{26}i$.

3 Modulus and Argument

For $z = a + bi$:

- **Modulus**: $|z| = \sqrt{a^2 + b^2}$
- **Argument**: $\theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$, adjusted by quadrant.

Example 1 (Hard)

If $|z-1| = |z+i|$, find $\arg(z)$.
Solution: Let $z = x + yi$. Then:

$$\sqrt{(x-1)^2 + y^2} = \sqrt{x^2 + (y+1)^2} \implies y = -x$$

Thus, $z = x(1-i)$, and $\arg(z) = \tan^{-1}(-1) = -\frac{\pi}{4}$.
Answer: $-\frac{\pi}{4}$.

Example 2 (Hard)

If $|z-2i| = |z+2i|$, find $\arg(z)$.
Solution: Let $z = x + yi$. Then:

$$\sqrt{x^2 + (y-2)^2} = \sqrt{x^2 + (y+2)^2} \implies x^2 + y^2 - 4y + 4 = x^2 + y^2 + 4y + 4 \implies y = 0$$

Thus, $z = x$, and $\arg(z) = 0$.
Answer: 0.

Example 3 (Hard)

Find $\arg(z)$ if $|z-3| = |z-i|$.
Solution: Let $z = x + yi$. Then:

$$\sqrt{(x-3)^2 + y^2} = \sqrt{x^2 + (y-1)^2} \implies x^2 - 6x + 9 + y^2 = x^2 + y^2 - 2y + 1 \implies x + y = 2$$

Thus, $z = x + (2-x)i$, $\arg(z) = \tan^{-1}\left(\frac{2-x}{x}\right)$. For $x = 1$, $\arg(z) = \tan^{-1}(1) = \frac{\pi}{4}$.
Answer: $\frac{\pi}{4}$ (for $x = 1$).

Example 4 (Hard)

If $|z| = 3$, find $\arg(z+3)$.
Solution: Let $z = 3e^{i\theta}$. Then $z+3 = 3(e^{i\theta}+1) = 3(1+\cos\theta + i\sin\theta)$. The argument is:

$$\arg(z+3) = \tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

Answer: $\frac{\arg(z)}{2}$.

Example 5 (Hard)

If $|z-1-i| = \sqrt{2}$, find possible $\arg(z)$.
Solution: Let $z = x+yi$. Then:

$$(x-1)^2 + (y-1)^2 = 2 \implies z \text{ lies on a circle centered at } (1,1), \text{ radius } \sqrt{2}$$

For $z = 2+i$, $\arg(z) = \tan^{-1}\left(\frac{1}{2}\right)$.
Answer: $\tan^{-1}\left(\frac{1}{2}\right)$ (one possible value).

4 Polar and Exponential Form

Polar form: $z = r(\cos\theta + i\sin\theta)$, where $r = |z|, \theta = \arg(z)$.
Exponential form: $z = re^{i\theta}$.

Example 1 (Hard)

Express $z = \frac{1+i\sqrt{3}}{1-i}$ in exponential form.
Solution:

$$z = \frac{1-\sqrt{3}}{2} + \frac{1+\sqrt{3}}{2}i, \quad r = \sqrt{2}, \quad \theta \approx \frac{2\pi}{3}$$

Answer: $\sqrt{2}e^{i\frac{2\pi}{3}}$.

Example 2 (Hard)

Express $z = \frac{1-i}{1+i}$ in exponential form.
Solution:

$$z = \frac{(1-i)(1-i)}{2} = \frac{1-2i-1}{2} = -i, \quad r = 1, \quad \theta = -\frac{\pi}{2}$$

Answer: $e^{-i\frac{\pi}{2}}$.

Example 3 (Hard)

Express $z = \frac{2+2i}{\sqrt{3}-i}$ in exponential form.
Solution:

$$z = \frac{(2+2i)(\sqrt{3}+i)}{4} = \frac{2\sqrt{3}+2i+2i\sqrt{3}-2}{4} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i, \quad r = \sqrt{2}, \quad \theta = \frac{3\pi}{4}$$

Answer: $\sqrt{2}e^{i\frac{3\pi}{4}}$.

Example 4 (Hard)

Express $z = -1 + i\sqrt{3}$ in exponential form.
Solution:

$$r = \sqrt{1+3} = 2, \quad \theta = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

Answer: $2e^{i\frac{2\pi}{3}}$.

Example 5 (Hard)

Express $z = \frac{i}{1-i}$ in exponential form.
Solution:

$$z = \frac{i(1+i)}{2} = \frac{i-1}{2} = -\frac{1}{2} + \frac{1}{2}i, \quad r = \sqrt{\frac{1}{2}}, \quad \theta = \frac{3\pi}{4}$$

Answer: $\frac{\sqrt{2}}{2}e^{i\frac{3\pi}{4}}$.

5 De Moivre's Theorem

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 1 (Hard)

Find $\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^5$.
Solution: Convert to polar form, apply De Moivres theorem:

$$e^{i\frac{5\pi}{3}} = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Answer: $\frac{1}{2} - \frac{i\sqrt{3}}{2}$.

Example 2 (Hard)

Find $(1+i)^8$.
Solution: Polar form: $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$. Then:

$$(\sqrt{2}e^{i\frac{\pi}{4}})^8 = 16e^{i2\pi} = 16$$

Answer: 16.

Example 3 (Hard)

Find $\left(\frac{1-i}{1+i}\right)^4$.
Solution: $\frac{1-i}{1+i} = -i$, so $(-i)^4 = (-1)^2 i^4 = 1$.
Answer: 1.

Example 4 (Hard)

Find $(-1+i\sqrt{3})^3$.
Solution: Polar form: $r=2$, $\theta=\frac{2\pi}{3}$. Then:

$$(2e^{i\frac{2\pi}{3}})^3 = 8e^{i2\pi} = 8$$

Answer: 8.

Example 5 (Hard)

Find $\left(\frac{1+i\sqrt{3}}{2}\right)^6$.
Solution: Polar form: $r=1$, $\theta=\frac{\pi}{3}$. Then:

$$(e^{i\frac{\pi}{3}})^6 = e^{i2\pi} = 1$$

Answer: 1.

6 Roots of Complex Numbers

The n-th roots of $z = r(\cos\theta + i\sin\theta)$:

$$\sqrt[n]{z} = r^{1/n}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1$$

Example 1 (Hard)

Find the fourth roots of -16.
Solution: $-16 = 16(\cos\pi + i\sin\pi)$. Roots are:

$$\pm\sqrt{2} \pm \sqrt{2}i$$

Answer: $\pm\sqrt{2} \pm \sqrt{2}i$.

Example 2 (Hard)

Find the cube roots of $8i$.
Solution: $8i = 8e^{i\pi/2}$. Roots are:

$$2e^{i(\pi/6+2k\pi/3)}, \quad k = 0, 1, 2 \implies \sqrt{3}+i, -\sqrt{3}+i, -2i$$

Answer: $\sqrt{3}+i, -\sqrt{3}+i, -2i$.

Example 3 (Hard)

Find the fifth roots of 1.
Solution: $1 = e^{i0}$. Roots are:

$$e^{i2k\pi/5}, \quad k = 0, 1, 2, 3, 4 \implies 1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}$$

Answer: $1, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.

Example 4 (Hard)

Find the square roots of -4.
Solution: $-4 = 4e^{i\pi}$. Roots are:

$$2e^{i(\pi/2+k\pi)}, \quad k = 0, 1 \implies 2i, -2i$$

Answer: $\pm 2i$.

Example 5 (Hard)

Find the sixth roots of -1.
Solution: $-1 = e^{i\pi}$. Roots are:

$$e^{i(\pi/6+k\pi/3)}, \quad k = 0, 1, 2, 3, 4, 5 \implies \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}, \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}, \ldots$$

Answer: $\frac{\sqrt{3}}{2} + \frac{i}{2}, i, -\frac{\sqrt{3}}{2} + \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}, -i, \frac{\sqrt{3}}{2} - \frac{i}{2}$.

7 Properties of Complex Numbers

- **Conjugate**: $\bar{z} = a - bi, z\bar{z} = |z|^2$.
- **Modulus**: $|z_1 z_2| = |z_1||z_2|, \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.
- **Argument**: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \bmod 2\pi$.

Example 1 (Hard)

If $|z| = 2$, find the minimum value of $|z - 3 - 4i|$.
Solution: Minimum distance is 3.
Answer: 3.

Example 2 (Hard)

If $|z| = 1$, find the maximum value of $|z + 2|$.
Solution: $z = e^{i\theta}$, so $|z + 2| = |e^{i\theta} + 2| \leq 1 + 2 = 3$.
Answer: 3.

Example 3 (Hard)

If $|z - 1| = 2$, find the minimum $|z|$.
Solution: Circle centered at $(1, 0)$, radius 2. Minimum $|z|$ at $z = -1$, $|z| = 1$.
Answer: 1.

Example 4 (Hard)

Find $\arg(z\bar{z})$ for any $z \neq 0$.
Solution: $z\bar{z} = |z|^2$, a positive real number, so $\arg(z\bar{z}) = 0$.
Answer: 0.

Example 5 (Hard)

If $|z| = 4$, find the maximum $|z - 5i|$.
Solution: Maximum distance at $\theta = \frac{\pi}{2}$, $|z - 5i| = 4 + 5 = 9$.
Answer: 9.

8 Solving Equations Involving Complex Numbers

Solve by equating parts or using polar forms.

Example 1 (Hard)

Solve $z^4 + 4z^2 + 16 = 0$.
Solution: Roots are $\pm(1 \pm i\sqrt{3})$.
Answer: $\pm 1 \pm i\sqrt{3}$.

Example 2 (Hard)

Solve $z^3 + 8 = 0$.
Solution: $z^3 = -8 = 8e^{i\pi}$. Roots:

$$2e^{i(\pi/3 + 2k\pi/3)}, \quad k = 0, 1, 2 \implies 1 + i\sqrt{3}, 1 - i\sqrt{3}, -2$$

Answer: $1 \pm i\sqrt{3}, -2$.

Example 3 (Hard)

Solve $z^2 - (2 + 2i)z + 4i = 0$.
Solution: Quadratic formula:

$$z = \frac{2 + 2i \pm \sqrt{(2 + 2i)^2 - 16i}}{2} = 1 + i \pm \sqrt{-2i}$$

Solve for roots numerically or polar form.
Answer: Approximate solutions.

Example 4 (Hard)

Solve $z^5 = 1$.
Solution: Roots are $e^{i2k\pi/5}$, $k = 0, 1, 2, 3, 4$.
Answer: $1, \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}, \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$.

Example 5 (Hard)

Solve $z^2 + z + 1 = 0$.
Solution: Roots:

$$z = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

Answer: $-\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$.

9 Complex Number as a Geometric Entity

Complex numbers are points in the Argand plane. Multiplication by $e^{i\theta}$ rotates by θ.

Example 1 (Hard)

If $|z| = 1$, find the locus of $w = \frac{z+1}{z-1}$.
Solution: Locus is the imaginary axis.
Answer: Imaginary axis.

Example 2 (Hard)

Find the locus of z if $|z - 1| = 2$.
Solution: Circle centered at $(1, 0)$, radius 2.
Answer: Circle, center $(1, 0)$, radius 2.

Example 3 (Hard)

Find the locus of $w = z^2$ if $|z| = 1$.
Solution: $z = e^{i\theta}$, $w = e^{i2\theta}$, so $|w| = 1$.
Answer: Unit circle.

Example 4 (Hard)

Find the locus of z if $\arg(z - i) = \frac{\pi}{4}$.
Solution: Ray from $(0, 1)$ at angle $\frac{\pi}{4}$.
Answer: Ray at $\frac{\pi}{4}$ from $(0, 1)$.

Example 5 (Hard)

Find the locus of $w = \frac{z-i}{z+i}$ if $|z| = 1$.
Solution: w maps to the real axis.
Answer: Real axis.

10 Applications of Complex Numbers

- **Electrical circuits**: Impedance as $R + Xi$.
- **Signal processing**: Fourier transforms.
- **Geometry**: Rotations and scaling.

Example 1 (Hard)

Find the image of the line $y = x$ under $w = z^2$.
Solution: Positive imaginary axis.
Answer: Positive imaginary axis.

Example 2 (Hard)

Find the image of $|z| = 1$ under $w = z + 1$.
Solution: Circle centered at $(1, 0)$, radius 1.
Answer: Circle, center $(1, 0)$, radius 1.

Example 3 (Hard)

Find the image of $y = 1$ under $w = z^2$.
Solution: $z = x + i$, $w = (x + i)^2 = x^2 - 1 + 2xi$, a parabola $u = v^2 - 1$.
Answer: Parabola $u = v^2 - 1$.

Example 4 (Hard)

Find the rotation angle of $w = (1 + i)z$.
Solution: $1 + i = \sqrt{2}e^{i\pi/4}$, rotates by $\frac{\pi}{4}$.
Answer: $\frac{\pi}{4}$.

Example 5 (Hard)

Find the image of $x = 1$ under $w = z + i$.
Solution: $z = 1 + yi$, $w = 1 + (y + 1)i$, a line $u = 1$.
Answer: Line $u = 1$.

11 Pattern-Based Exercises (Hard)

11.1 Exercise 1

Solve $z^3 = \bar{z}$.
Answer: $0, \pm 1, \pm i$.

11.2 Exercise 2

Find the minimum $|z|$ if $|z - 2i| + |z + 2i| = 8$.
Answer: $2\sqrt{3}$.

11.3 Exercise 3

Find $\left(\frac{1+i}{1-i}\right)^{100}$.
Answer: 1.

11.4 Exercise 4

Solve $z^4 = i$.
Answer: $\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}, \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}, \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}, \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$.

11.5 Exercise 5

Find the maximum $|z|$ if $|z-1|+|z+1|=4$.
Answer: 2.

12 Multiple Choice Questions (Hard)

1. If $|z|=2$, the maximum value of $|z-4|$ is:
   (a) 2
   (b) 4
   (c) 6
   (d) 8

   Answer: (c) 6.

13 Integer-Type Questions

1. Number of distinct solutions to $z^5=-1$: **Answer**: 5.
2. Real part of $\left(\frac{1+i\sqrt{3}}{2}\right)^3$: **Answer**: $-\frac{1}{2}$.

14 Assertion-Reason Questions

1. **Assertion**: If $|z|=1$, then $\arg(z^2)=2\arg(z)$.
   Reason: $\arg(z^n)=n\arg(z) \bmod 2\pi$.
   Answer: Both true, Reason explains Assertion.

15 Summary

Complex numbers are critical for JEE Main, with applications in algebra, geometry, and physics. Key formulas:

$$z^n = r^n e^{in\theta}, \quad \sqrt[n]{z} = r^{1/n} e^{i\frac{\theta+2k\pi}{n}}, \quad |z_1 z_2| = |z_1||z_2|$$

16 Additional Hard Exercises

1. Solve $z^4=-i$. **Answer**: $\pm\frac{\sqrt{2}}{2}(1+i), \pm\frac{\sqrt{2}}{2}(1-i)$.
2. Find the locus of z if $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{2}$. **Answer**: Circle centered at $(0,0)$, radius 1, excluding $z=-1$.